

**MULTI-LEVEL MODULATION CODES
AND
MULTI-STAGE DECODING**

Technical Report

to

NASA

Goddard Space Flight Center

Greenbelt, Maryland 20771

Grant Number NAG 5-931

Report Number NASA 90-002

Shu Lin

Principal Investigator

Department of Electrical Engineering

University of Hawaii at Manoa

Honolulu, Hawaii 96822

February 28, 1990

(NASA-CR-186376) MULTI-LEVEL MODULATION
CODES AND MULTI-STAGE DECODING (Hawaii
Univ.) 38 p CSCL 17B

N91-11953

Unclas
0269638

63/32

*GODDARD
GRANT
1N-32-CR*

269638

388

Introduction

- Multi-level method is a powerful technique for constructing bandwidth efficient modulation (or signal space) codes. It allows us to construct modulation codes systematically with ^{the construction of} arbitrary large minimum squared Euclidean distance from component codes (binary or nonbinary) in conjunction with proper bits-to-signal mapping.
- If the component codes are chosen properly, the resultant modulation code not only has good minimum squared Euclidean distance but is also rich in structural (algebraic and geometric) properties such as: linear structure, phase invariant property and trellis structure.

- A modulation code with linear structure has invariant (Euclidean) distance distribution, i.e., the distance distribution is the same for all code sequences. As a result of this distance symmetry property, the error probability over a Gaussian channel does not depend on which code sequence is transmitted. Linearity also simplifies the encoding and decoding implementations.
- Phase invariant (or phase symmetry) property is useful in resolving carrier-phase ambiguity and ensuring rapid carrier-phase resynchronization after temporary loss of synchronization. It is desirable for a modulation code to have as much phase symmetry as possible.
- If the component codes have trellis structure, the resultant multi-level modulation code also has trellis structure. A trellis diagram for a multi-level modulation code can be obtained by taking the direct product of the trellis diagrams for its component codes.
- Trellis structure allows us to decode a multi-level modulation code with the soft-decision Viterbi decoding algorithm.

- Furthermore, the multi-level structure allows us to decode a multi-level modulation code with the multi-stage decoding, i.e., component codes are decoded sequentially stage by stage, decoding information is passed from one stage to the next. This type of decoding reduces the decoding complexity. Multi-stage decoding is not optimum even though the decoding of each component is optimum. In this case, it is a suboptimum decoding. Based on ^{the} our analysis and simulation results, the difference in error performance between the optimum decoding of the overall multi-level modulation code and the suboptimum multi-stage decoding of the code is very little, a fraction of dB loss.

Multi-Level Code Construction

- **Construction Steps**

(1) **Selection of a signal set:** A set of 2^ℓ signal points.

(2) **Signal labeling:** Each signal point is labeled by a string of ℓ bits. Such labeling is said to have ℓ levels. Labeling is generally done by a set partitioning process.

(3) **Selection of component codes:** The component codes may be binary or nonbinary, block or trellis.

(4) **Code construction:** Combine component codes into a multi-level code.

(5) **Bits-to-signal mapping:** Map a label into a signal point. This mapping results in a multi-level modulation code.

- If the number of component codes is equal to the number of labeling levels (ℓ), the resultant modulation code is called a **basic multi-level modulation code**.

Example

An 8-PSK 3-Level Block Modulation Code

Signal Set

- Choose an 8-PSK signal set S as shown in Figure 1.

Labeling

- Label each of the 8 signal points by a string of 3 bits,

$$a b c$$

where a is called the first-level label and c is the third level label.

- The labeling is achieved by set partitioning process as shown in Figure 2. The signal set is partitioned into a chain of partitions. The first partition consists of two disjoint subsets which are labeled by “0” and “1”. The second partition consists of four disjoint subsets which are labeled by 00,01,10 and 11 respectively. The third partition consists of 8 disjoint subsets, each consisting of only one signal point, which are labeled by 8 unique 3-tuples.

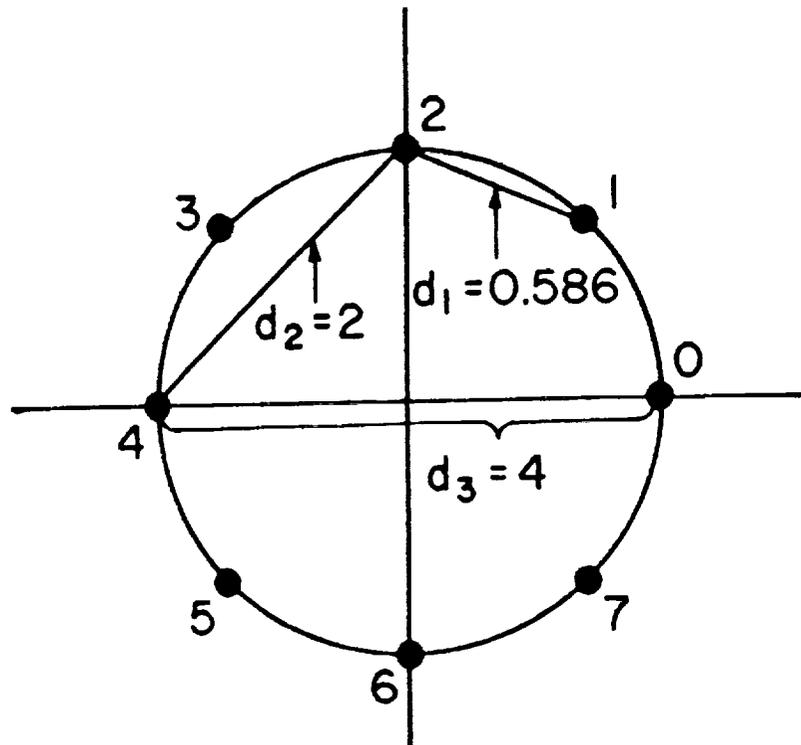


Figure 1. An 8-PSK signal set.

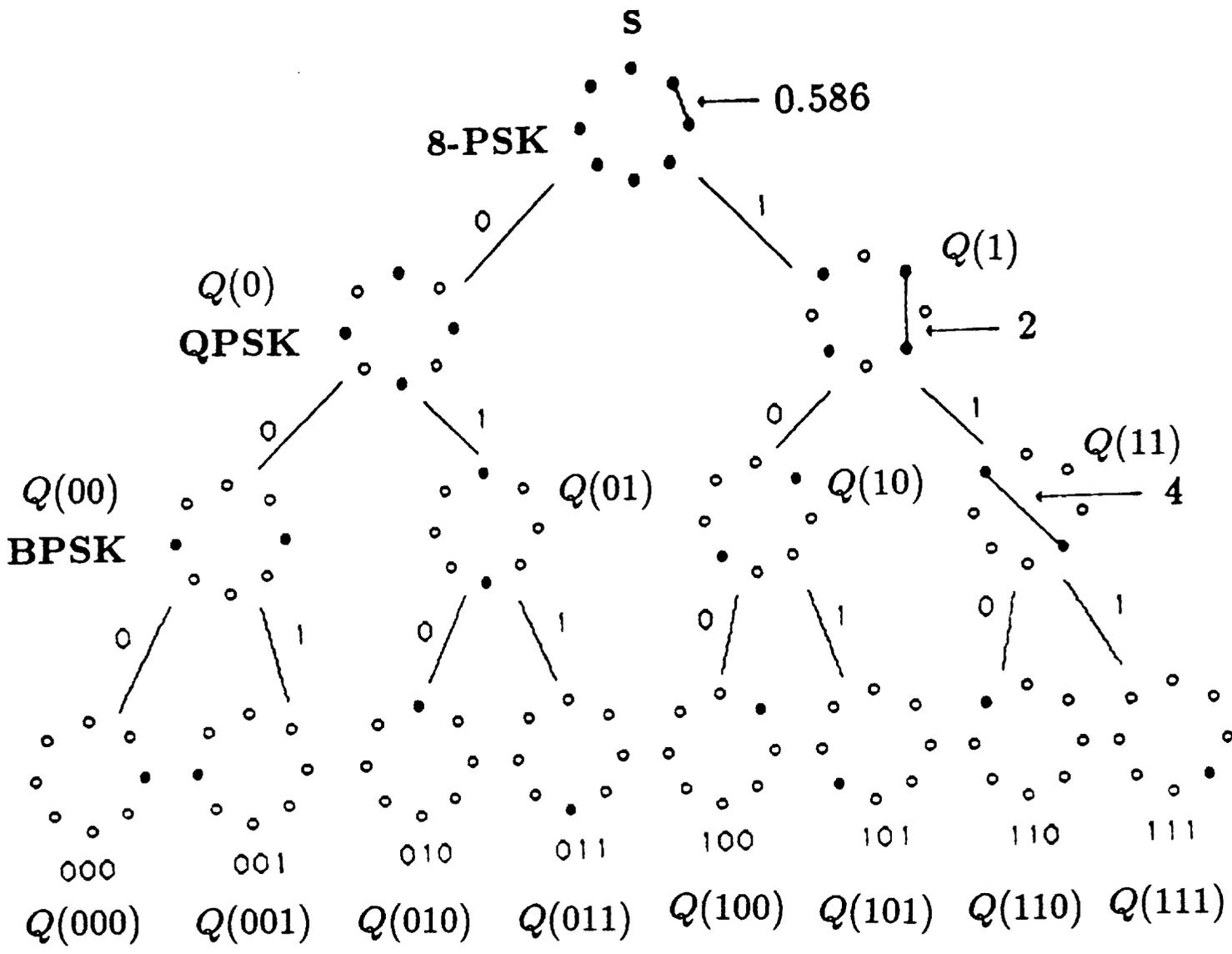


Figure 2. Signal labeling and 8-PSK/QPSK/BPSK partitioning chain.

- The partition is carried out in such a way that, as the partition level increases, the **intra-set distance** (the minimum squared Euclidean distance among signal points) of a set in a partition increases. For our example, the intra-set distances at 3 partition levels are:

0.586, 2, 4,

respectively.

- From Figure 2, we see that each subset in the first partition is a QPSK signal set and each subset in the second partition is a BPSK.
- Let $\lambda(\cdot)$ denote the mapping defined by the labeling. Then $\lambda(abc) = s$ is a signal point in the signal set S .

- The 8 signal points and their corresponding labels are shown in Figure 3.
- Each prefix of a label represents a subset of signal points in S ,

$$\begin{array}{lcl}
 a & \iff & \left\{ \begin{array}{l} 4 \text{ signal points in a} \\ \text{QPSK signal set} \end{array} \right\} \\
 ab & \iff & \left\{ \begin{array}{l} 2 \text{ signal points in a} \\ \text{BPSK signal set} \end{array} \right\} \\
 abc & \iff & \left\{ \begin{array}{l} \text{a single point in the} \\ \text{8-PSK signal set} \end{array} \right\}
 \end{array}$$

- Let $Q(a)$ denote the set of signal points whose labels have “ a ” as the prefix ($Q(a) = \text{QPSK}$).
- Let $Q(ab)$ denote the set of signal points whose labels have “ ab ” as the prefix ($Q(ab) = \text{BPSK}$).
- Let d_1, d_2 , and d_3 be the intra-set distances of S , $Q(a)$ and $Q(ab)$ respectively. For our example,

$$d_1 = 0.586, \quad d_2 = 2, \quad d_3 = 4.$$

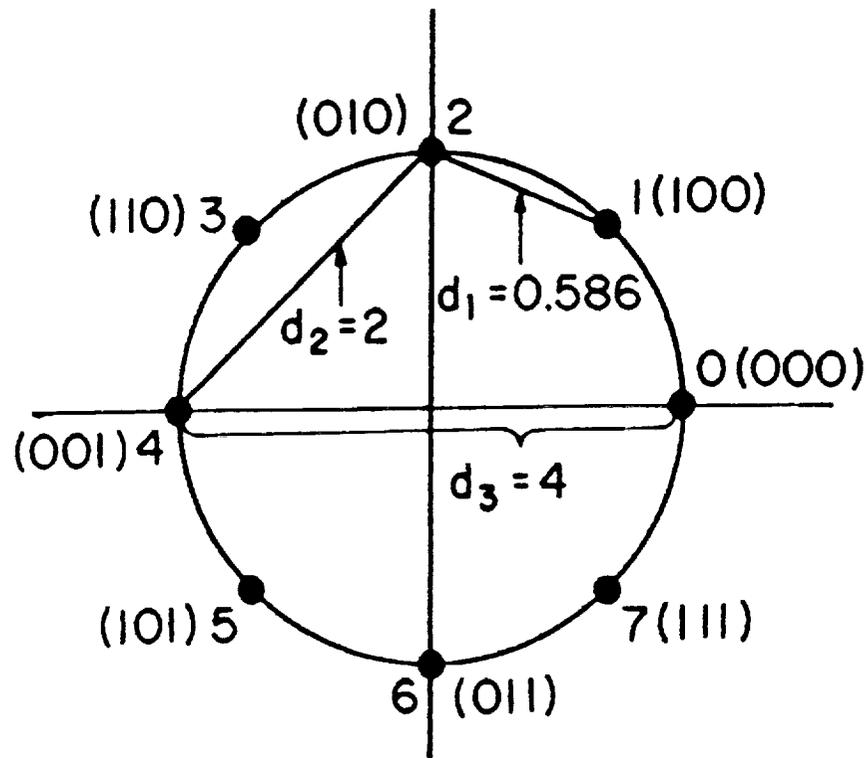
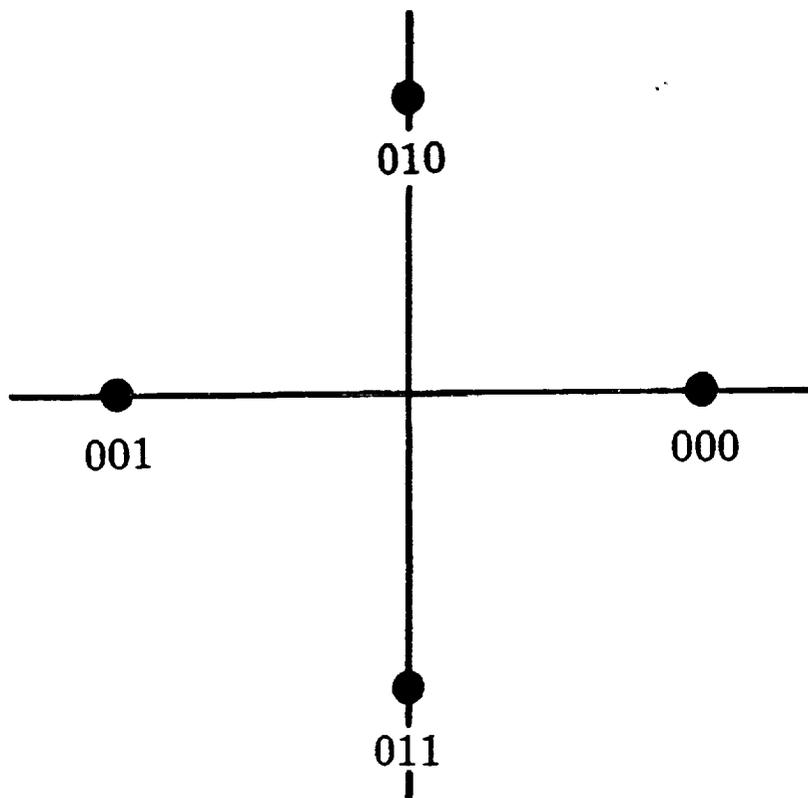


Figure 3. 8-PSK signal points and their labels.

$Q(0)$
QPSK



$Q(1)$
QPSK

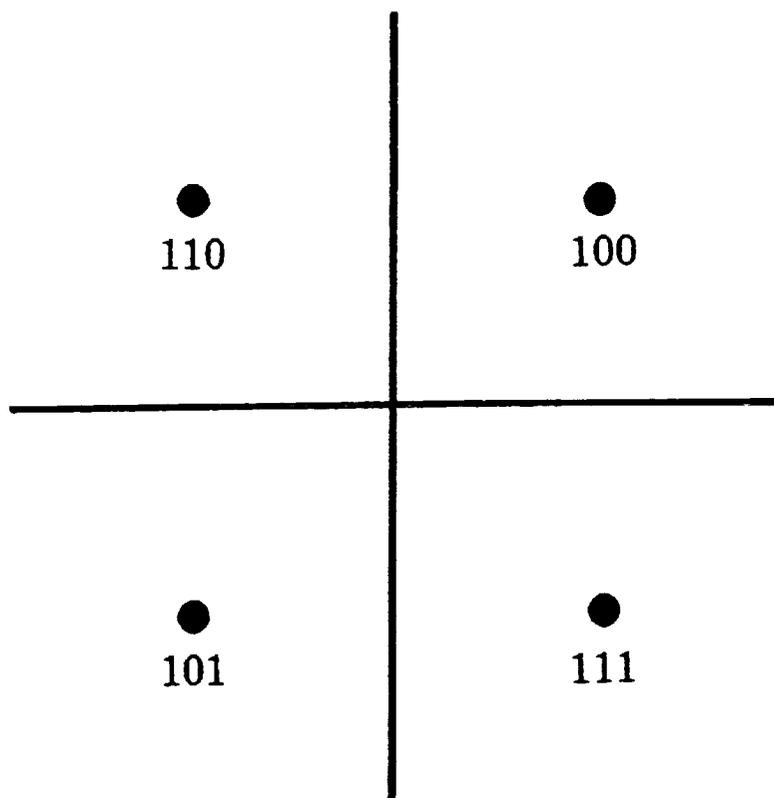


Figure 4. QPSK signal constellations.

Selection of Component Codes

- For $1 \leq i \leq 3$, let C_i be a binary (n, k_i) code with minimum Hamming distance δ_i .

Code Construction

- Let

$$\bar{a} = (a_1, a_2, \dots, a_i, \dots, a_n)$$

$$\bar{b} = (b_1, b_2, \dots, b_i, \dots, b_n)$$

$$\bar{c} = (c_1, c_2, \dots, c_i, \dots, c_n)$$

be three codewords in C_1 , C_2 and C_3 respectively.

- Form the following sequence,

$$\bar{a} * \bar{b} * \bar{c} \triangleq (a_1 b_1 c_1, a_2 b_2 c_2, \dots, a_n b_n c_n).$$

- For $1 \leq i \leq n$, we take $a_i b_i c_i$ as the label of a signal point in the 8-PSK signal set. Then

$$\lambda(\bar{a} * \bar{b} * \bar{c}) = (\lambda(a_1 b_1 c_1), \lambda(a_2 b_2 c_2), \dots, \lambda(a_n b_n c_n))$$

is a sequence of n 8-PSK signals.

- The set

$$\begin{aligned} C &\triangleq C_1 * C_2 * C_3 \\ &= \{ \lambda(\bar{a} * \bar{b} * \bar{c}) : \bar{a} \in C_1, \bar{b} \in C_2 \text{ and } \bar{c} \in C_3 \} \end{aligned}$$

is a basic 3-level block 8-PSK modulation code.

Minimum Squared Euclidean Distance

- The minimum squared Euclidean distance of a basic 3-level modulation code is

$$D[C] = \min \{ \delta_1 d_1, \delta_2 d_2, \delta_3 d_3 \}$$

- For our example,

$$D[C] = \min \{ 0.586\delta_1, 2\delta_2, 4\delta_3 \}$$

Remark

- In the above construction, each component code contributes one level of labeling.

Soft-Decision Multi-Stage Decoding

- Component codes are decoded with soft-decision maximum likelihood decoding, one at a time stage-by-stage. The decoder information at each stage is passed to the next stage. The decoding process begins with the first-level component code and ends at the last-level component code.
- Assume that the channel is an AWGN channel.
- Let

$$\bar{r} = (r_1, r_2, \dots, r_i, \dots, r_n)$$

be the received sequence at the output of the demodulator where

$$r_i = (x_i, y_i) \in R^2.$$

First Stage of Decoding

- Let

$$\bar{a} = (a_1, a_2, \dots, a_i, \dots, a_n)$$

be a code sequence in C_1 .

- Let $d[r_i, Q(a_i)]$ be the minimum squared Euclidean distance between r_i and the points in $Q(a_i)$. For our example, $Q(a_i)$ is either the QPSK signal set shown in Figure 4(a) or the QPSK signal set shown in Figure 4(b).
- For every codeword \bar{a} in C_1 , we compute the distance,

$$d(\bar{r}, \bar{a}) \triangleq \sum_{i=1}^n d[r_i, Q(a_i)].$$

- Decode \bar{r} into \bar{a} for which $d(\bar{r}, \bar{a})$ is the minimum.

Second Stage of Decoding

- The decoded information, \bar{a} , of the first stage is passed to the second stage.
- Let $d[r_i, Q(a_i b_i)]$ be the minimum squared Euclidean distance between r_i and the points in $Q(a_i b_i)$. For our example, $Q(a_i b_i)$ is a BPSK.
- For every codeword \bar{b} in C_2 , we compute,

$$d(\bar{r}, \bar{a} * \bar{b}) \triangleq \sum_{i=1}^n d[r_i, Q(a_i b_i)].$$

- Decode \bar{r} into \bar{b} for which $d(\bar{r}, \bar{a} * \bar{b})$ is the minimum.

Third Stage of Decoding

- The decoded information at the first and second stages, \bar{a} and \bar{b} , are made available to the third stage.
- For every codeword \bar{c} in C_3 , we compute,

$$\begin{aligned}d(\bar{r}, \bar{a} * \bar{b} * \bar{c}) &\triangleq \sum_{i=1}^n d[r_i, Q(a_i b_i c_i)] \\ &\triangleq \sum_{i=1}^n d[r_i, \lambda(a_i b_i c_i)].\end{aligned}$$

- Decode \bar{r} into \bar{c} for which $d(\bar{r}, \bar{a} * \bar{b} * \bar{c})$ is the minimum.
- $(\bar{a}, \bar{b}, \bar{c})$ forms the decoded set.

Remarks

- If each component has a trellis structure, then the Viterbi decoding algorithm can be applied to decode each component code.
- The multi-stage decoding algorithm (**MSD**) is not optimum even though the decoding of each component code is optimum. It is **suboptimum**.
- The difference in performance between the optimum decoding of the overall multi-level modulation code and the suboptimal MSD is very small, a fraction of dB in coding gain.
- **MSD reduces the decoding complexity drastically.**

A Specific Example

- Let $RM_{m,r}$ denote an r -th order Reed-Muller code of length $n = 2^m$ and minimum Hamming distance $\delta = 2^{m-r}$.
- $RM_{m,r}$ has a 4-section $2^{\binom{m-1}{r}}$ -state trellis.
- Let P_n denote the even weight single parity-check code of length n .
- Choose the 8-PSK as the signal set.
- Let $C_1 = RM_{5,1}$, $C_2 = RM_{5,3}$ and $C_3 = P_{32}$. Then $\delta_1 = 16$, $\delta_2 = 4$ and $\delta_3 = 2$.
- The code

$$C = RM_{5,1} * RM_{5,3} * P_{32}$$

is a basic 3-level 8-PSK modulation code of length 32. The minimum squared Euclidean distance is

$$\begin{aligned} D[C] &= \min \{0.586 \times 16, 2 \times 4, 4 \times 2\} \\ &= 8. \end{aligned}$$

- Each code sequence contains

$$\left\{ \binom{5}{0} + \binom{5}{1} \right\} \\ + \left\{ \binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} \right\} \\ + 31 = 63$$

information bits.

- The spectral efficiency of the code is

$$\eta[C] = \frac{63}{32} = 1.966 \text{ bits/symbol}$$

- The effective rate of the code is

$$R[C] = \frac{63}{64} \text{ bits/dimension}$$

- The first component code $C_1 = \text{RM}_{5,1}$ has a 4-section 16-state trellis; the second component code $C_2 = \text{RM}_{5,3}$ also has a 4-section 16-state trellis; and the third component code $C_3 = \text{P}_{32}$ has a 32-section 2-state trellis.
- The overall modulation code $C = C_1 * C_2 * C_3$ has a 512-state trellis.
- With MSD, each component code can be decoded with the soft-decision Viterbi decoding.
- Viterbi decoding of the overall modulation code is **rather complicated and expensive**.
- The error performance of the code with various decodings is shown in Figure 5. We see that, with soft-decision multi-stage decoding, there is almost 5 dB real coding gain over the uncoded QPSK at the block-error-rate (BER) 10^{-6} .
- The asymptotic coding gain of this code over the uncoded QPSK with optimal decoding is

$$D[C]_{asy} = 10 \log_{10} \frac{8}{2} = 6 \text{ dB}.$$

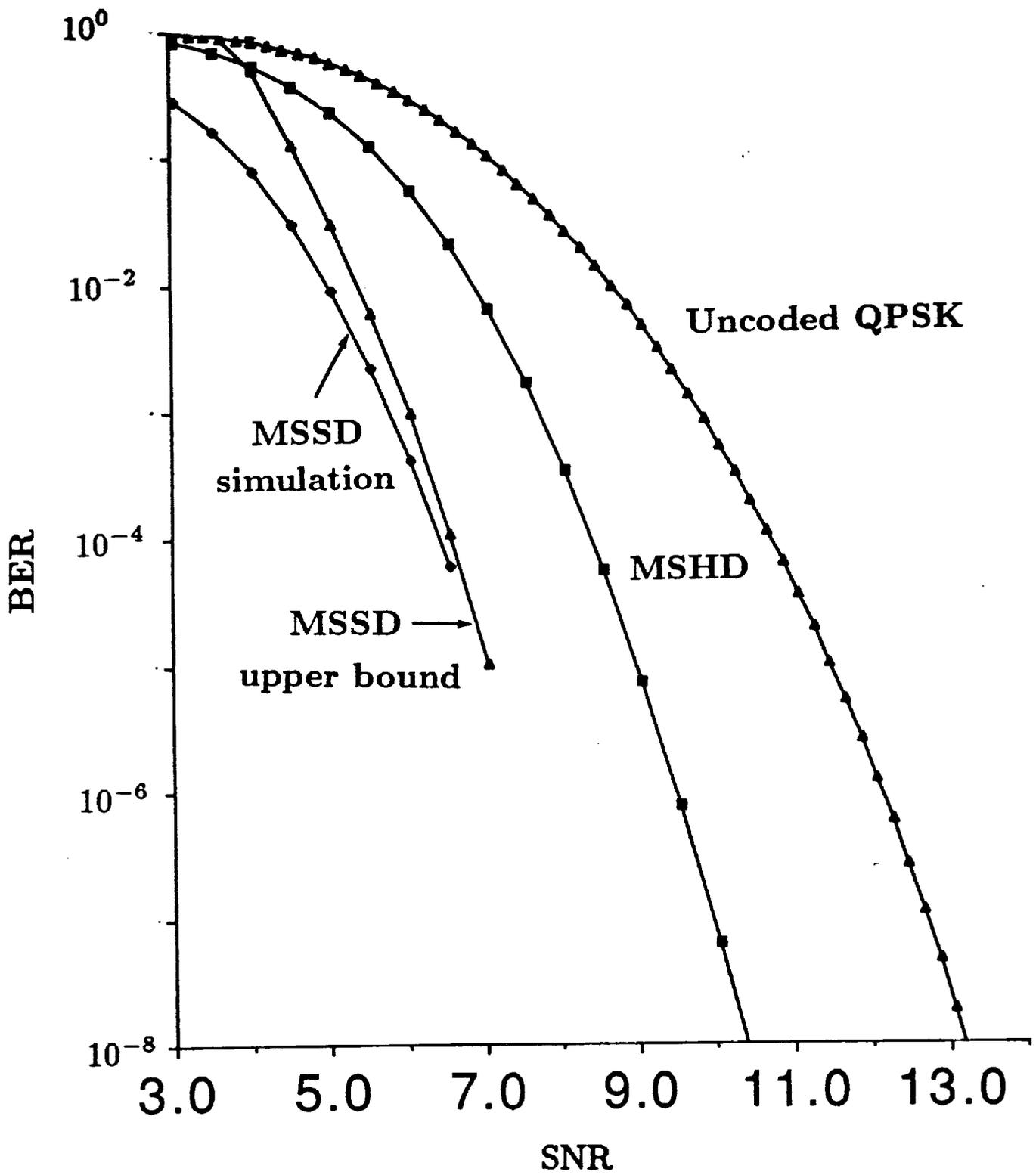


Figure 5. Error performance of the basic 3-level block 8-PSK modulation code $RM_{5,1} * RM_{5,3} * P_{32}$.

Hard-Decision Multi-Stage Decoding

- Let

$$\bar{r} = (r_1, r_2, \dots, r_i, \dots, r_n)$$

be the received sequence at the output of the demodulator, where r_i is a point in the R^2 -plane.

First Stage of Decoding

- Divide R^2 into two decision regions, R_0^2 and R_1^2 , where R_0^2 contains the signal points whose labels have “0” as the prefix and R_1^2 contains the signal points whose labels have “1” as the prefix. For our example of a 3-level 8-PSK modulation code, the division of R^2 is shown in Figure 6.
- **Hard decision:** If r_i is a point in R_0^2 , set the output of the first-stage detector,

$$z_i^{(1)} = 0.$$

If r_i is a point in R_1^2 , set the detector output

$$z_i^{(1)} = 1.$$

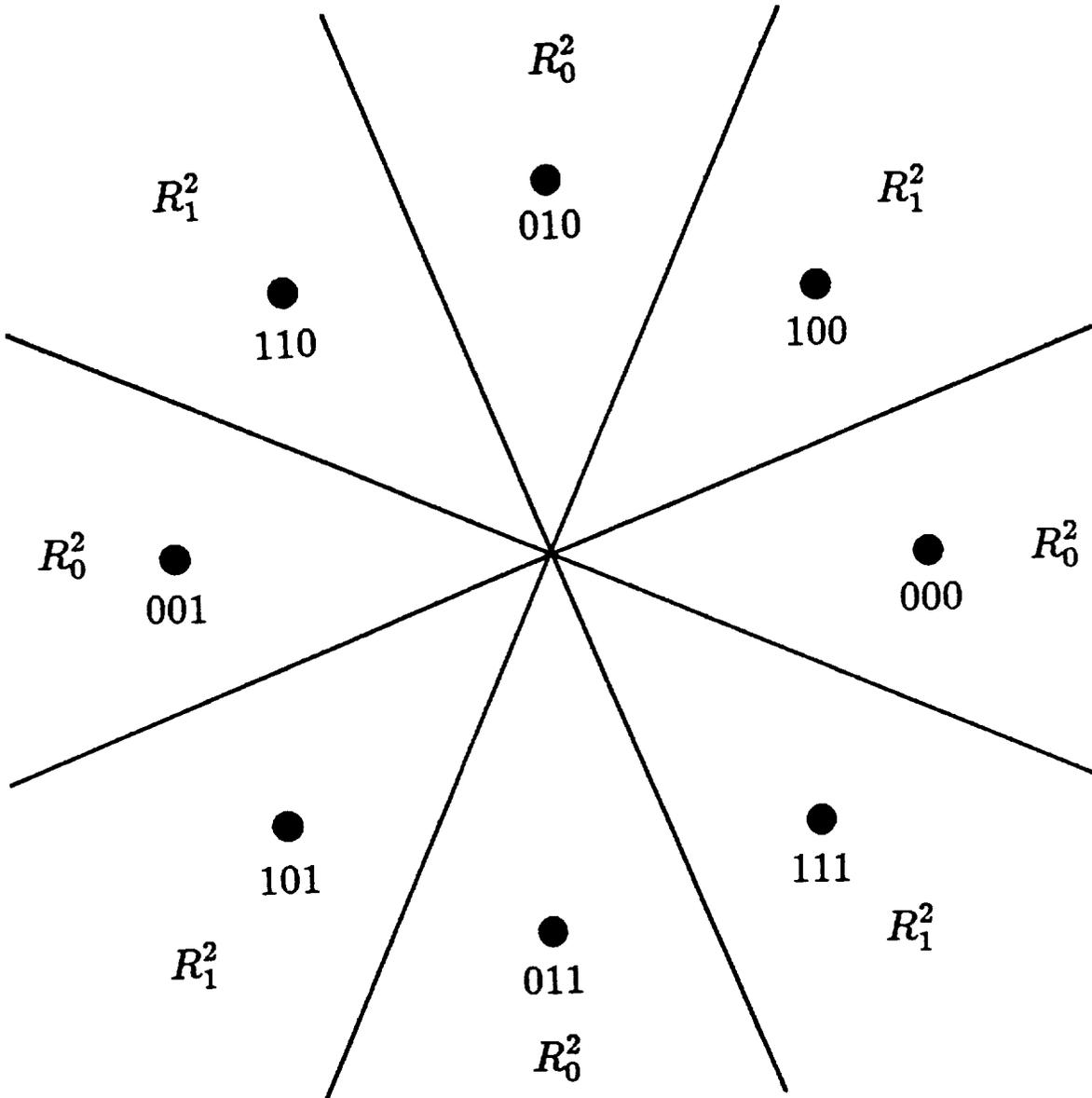


Figure 6. Decision regions for the first label bit.

- The binary vector,

$$\bar{z}^{(1)} = (z_1^{(1)}, z_2^{(1)}, \dots, z_i^{(1)}, \dots, z_n^{(1)}),$$

at the output of the detector is then decoded based on the first component code C_1 . The decoding may be maximum likelihood decoding or algebraic decoding. The schematic diagram of first-stage decoding is shown in Figure 7.

- Let

$$\bar{a} = (a_1, a_2, \dots, a_i, \dots, a_n)$$

be the decoded codeword.

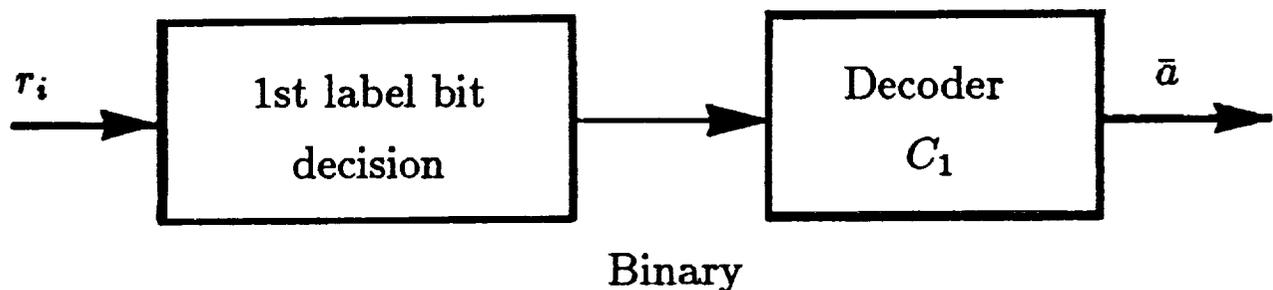


Figure 7. Schematic diagram of the first-stage hard-decision decoding.

Second Stage of Decoding

- The decoded information, \bar{a} , from the first decoding stage is passed to the second stage.
- For $a_i = 0$, divide the R^2 -plane into two decision regions, R_{00}^2 and R_{01}^2 , where R_{00}^2 contains those signal points whose labels have “00” as the prefix and R_{01}^2 contains those signal points whose labels have “01” as the prefix.
- For $a_i = 1$, the R^2 -plane is divided into two decision regions, R_{10}^2 and R_{11}^2 , where R_{10}^2 contains the signal points whose labels have “10” as the prefix and R_{11}^2 contains the signal points whose labels have “11” as the prefix.
- For our example, the divisions of R^2 -plane are shown in Figure 8.

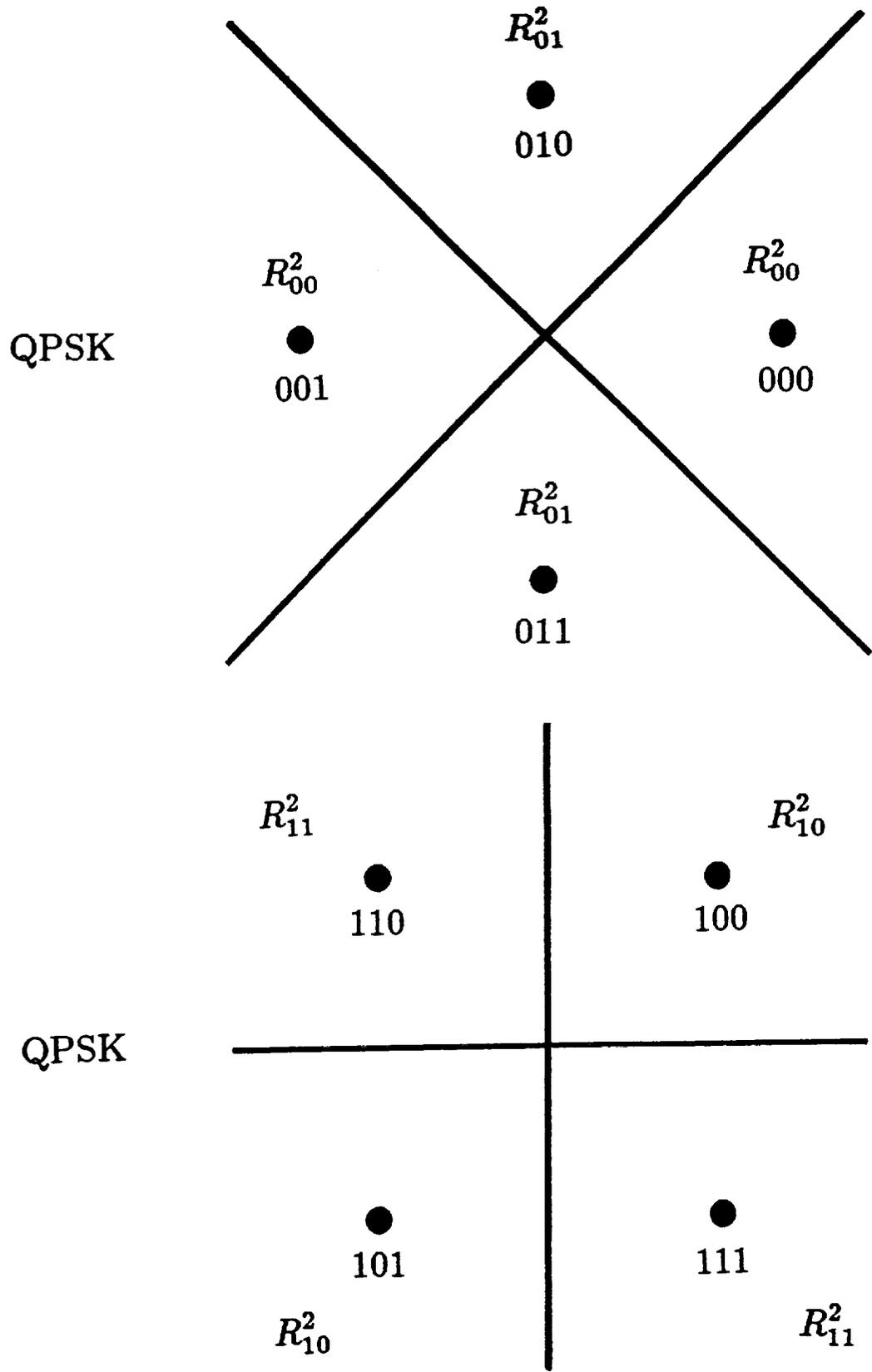


Figure 8. Decision regions for the second label bit.

- **Hard decision:** If $a_i = 0$ and $r_i \in R_{00}^2$, then set the output of the second stage detector,

$$z_i^{(2)} = 0.$$

If $a_i = 0$ and $r_i \in R_{01}^2$, then set the detector output,

$$z_i^{(2)} = 1.$$

If $a_i = 1$ and $r_i \in R_{10}^2$, then set the detector output,

$$z_i^{(2)} = 0.$$

If $a_i = 1$ and $r_i \in R_{11}^2$, then set the detector output,

$$z_i^{(2)} = 1.$$

- The binary vector,

$$\bar{z}^{(2)} = (z_1^{(2)}, z_2^{(2)}, \dots, z_i^{(2)}, \dots, z_n^{(2)}),$$

at the output of the detector is decoded based on the second component code C_2 .

- The schematic diagram of the second-stage decoder is shown in Figure 9.

- Let

$$\bar{b} = (b_1, b_2, \dots, b_i, \dots, b_m)$$

be the decoded codeword at the second-stage of the decoding.

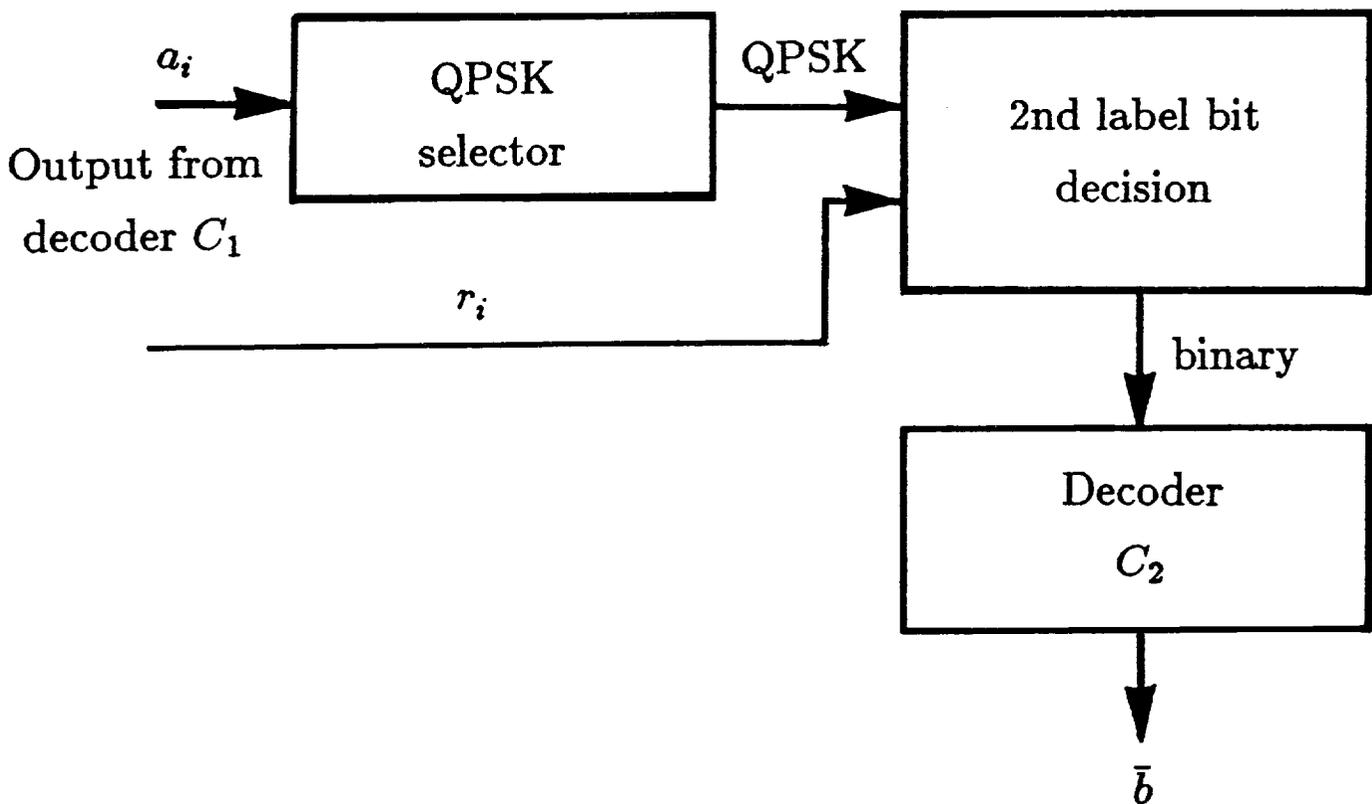


Figure 9. Schematic diagram of the second-stage hard-decision decoding.

Third Stage of Decoding

- The decoded information, \bar{a} and \bar{b} , at the first and second decoding stages is passed to the third stage.
- For $1 \leq i \leq n$, the R^2 -plane is divided into two decision regions, $R_{a_i, b_i, 0}^2$ and $R_{a_i, b_i, 1}^2$ based on a_i and b_i , where $R_{a_i, b_i, 0}^2$ contains the signal points whose labels have “ $a_i, b_i, 0$ ” as the prefix and $R_{a_i, b_i, 1}^2$ contains the signal points whose labels have “ $a_i, b_i, 1$ ” as the prefix.
- For our example, the divisions of the R^2 -plane are shown in Figure 10.

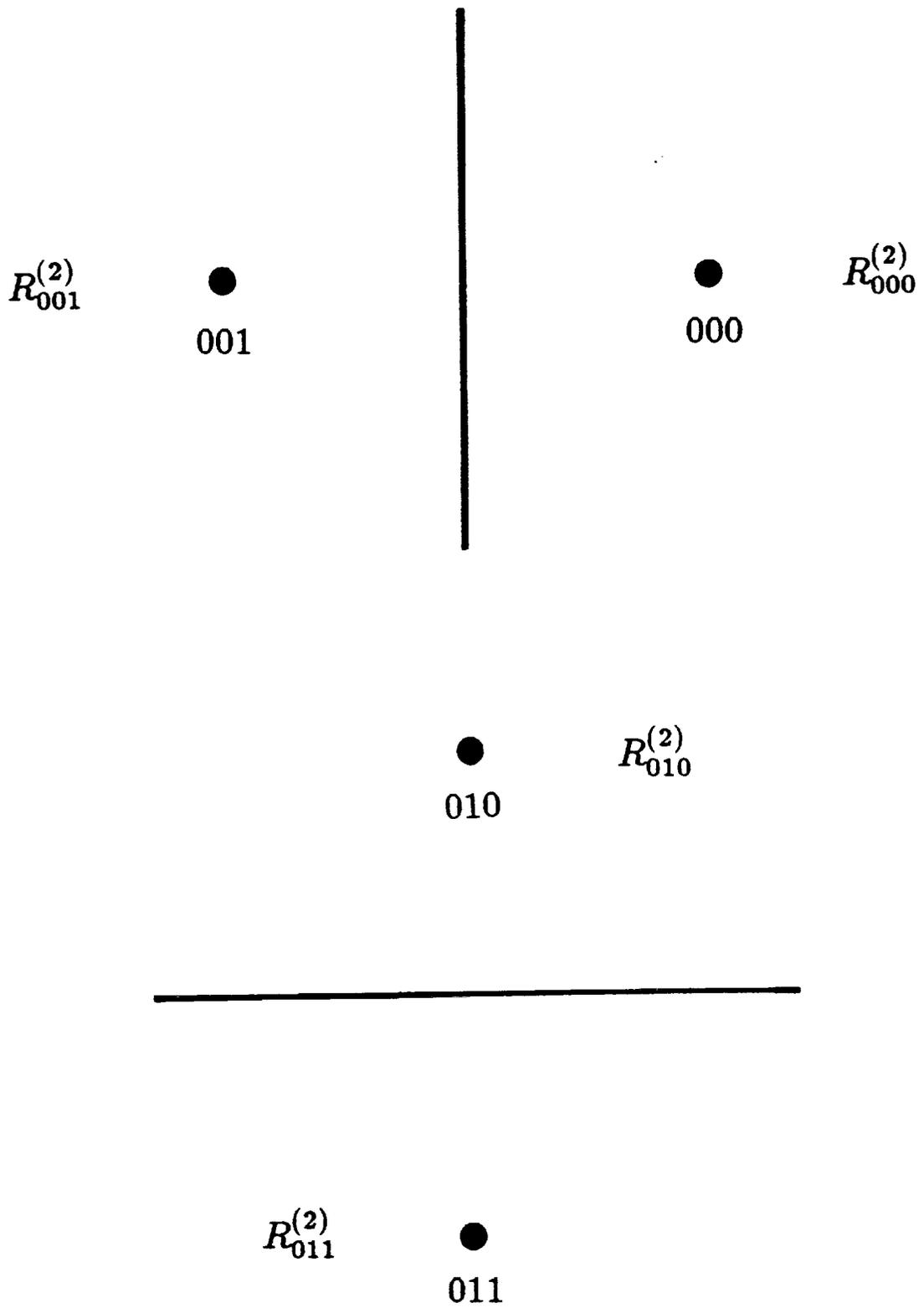


Figure 10. Decision regions for the third label bit.

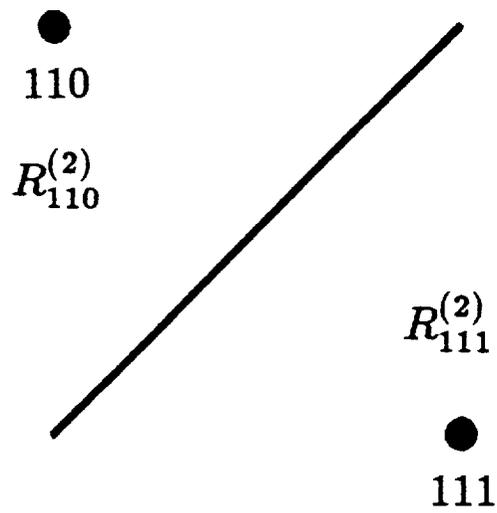
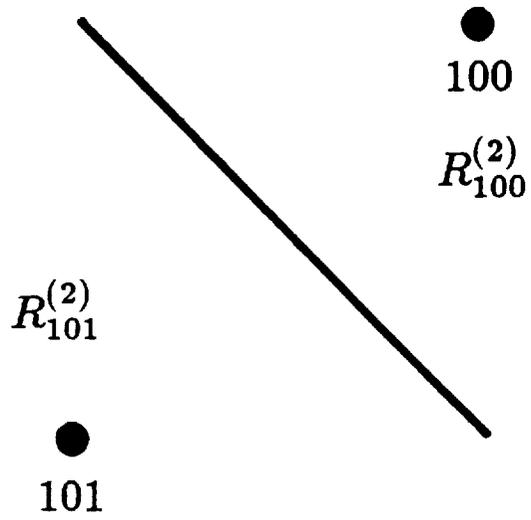


Figure 10. Decision regions for the third label bit.

- **Hard decision:** For given a_i, b_i , if r_i is a point in $R_{a_i, b_i, 0}^2$, then set the output of the third-stage detector,

$$z_i^{(3)} = 0;$$

otherwise, set

$$z_i^{(3)} = 1.$$

- The binary vector,

$$\bar{z}^{(3)} = (z_1^{(3)}, z_2^{(3)}, \dots, z_i^{(3)}, \dots, z_n^{(3)})$$

at the output of the third-stage detector is decoded based on the third component code C_3 .

- The schematic diagram of the third stage decoder is shown in Figure 11.
- Let

$$\bar{c} = (c_1, c_2, \dots, c_i, \dots, c_n)$$

be the decoded codeword in C_3 .

- If there are only 3 component codes, $(\bar{a}, \bar{b}, \bar{c})$ forms the decoded set.

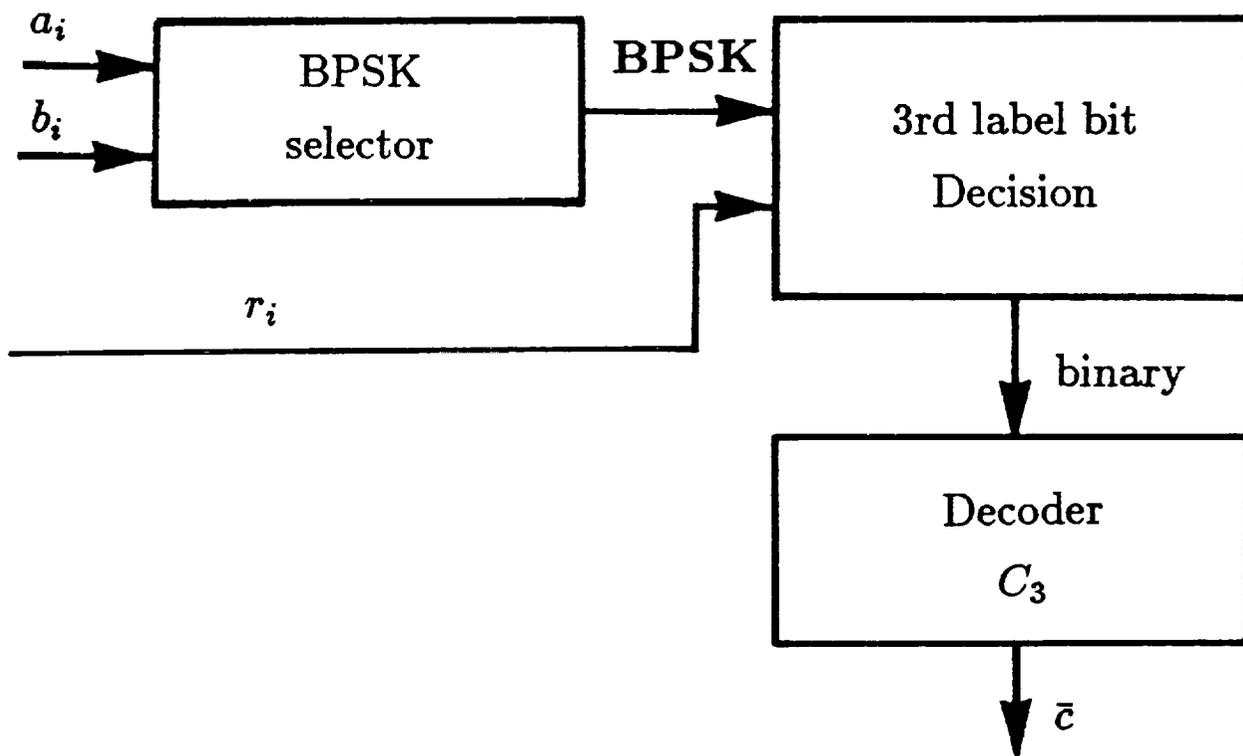


Figure 11. Schematic diagram of the third-stage hard-decision decoding.

A Specific Example

- Consider the basic 3-level 8-PSK modulation code,

$$RM_{5,1} * RM_{5,3} * P_{32}.$$

- With hard-decision MSD, the error performance of the code is shown in Figure 5. We see that there is a 2 dB loss compared with soft-decision MSD at $BER = 10^{-6}$. However, there is still a 2.8 dB coding gain over the uncoded QPSK at $BER = 10^{-6}$.
- Hard-decision MSD further reduces the decoding complexity of modulation codes while still maintains reasonable coding gain over the uncoded system.

Remarks

- MSD of multi-level modulation codes provides a good trade-off between complexity and performance.
- To achieve high effective rate (or spectral efficiency) and large gain over uncoded systems, we may use long powerful component codes.
- MSD of multi-level trellis modulation codes is similar to that of multi-level block modulation codes.